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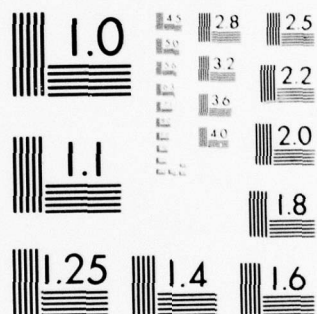
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TECHNICAL REPORT T-78-42

**MOMENT ESTIMATORS AND MAXIMUM LIKELIHOOD
ESTIMATORS FOR THE RICIAN/LAHA'S BESSEL
DISTRIBUTION: A COMPARATIVE STUDY**

Latricha Greene and Jerry Vickers
Aeroballistics Directorate
Technology Laboratory

February 1978



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CONTENTS

	Page
I. INTRODUCTION	3
II. BACKGROUND	3
III. EXPECTED VALUE AND VARIANCE OF \hat{A}^2 AND $\hat{\psi}_0$	4
IV. ASYMPTOTIC SAMPLING DISTRIBUTION OF \hat{A}^2 AND $\hat{\psi}_0$	8
V. COMPARISON OF MOMENT ESTIMATORS TO MAXIMUM LIKELIHOOD ESTIMATORS	19
VI. CONCLUSIONS	22
REFERENCES	25

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I. INTRODUCTION

The purpose of this report is to present approximations to the standard errors of moment estimators for the parameters of the Rician distribution and Laha's Bessel distribution. The asymptotic properties of these estimators will be investigated and a comparison to similar estimates using maximum likelihood techniques will be performed.

The findings in this report are an extension of efforts by this organization to develop a method to be employed in determining the frequency distribution of laser-radar return signals and to make inferences regarding future signal data based on this characterization.

II. BACKGROUND

In an earlier study [1] supporting research in the area of laser-radar technology, moment estimators for the Rician distribution and Laha's Bessel distribution were developed. The generalized Rician distribution can be given by

$$P(R) = \left(\frac{R}{\psi_0} \right) \exp \left\{ - \frac{(R^2 + A^2)}{2\psi_0} \right\} I_0 \left(\frac{RA}{\psi_0} \right) \quad (1)$$

where

$$R = \sqrt{x_1^2 + x_2^2} \geq 0 \quad ,$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ is a random normal vector,}$$

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \text{ is the mean vector,}$$

$$A = \sqrt{A_1^2 + A_2^2} \geq 0 \quad ,$$

M_2 is a positive definite covariance matrix $I \psi_0$ where ψ_0 is a positive constant, I is the identity matrix, and I_0 is the modified Bessel function of zero order.

By applying the transformation $X = R^2$, Laha's Bessel distribution was shown to be

$$P(X) = \left(\frac{1}{2\psi_0} \right) \exp \left\{ - \frac{(X - A^2)}{2\psi_0} \right\} I_0 \left(\frac{\sqrt{X} A}{\psi_0} \right) \quad (2)$$

The mean and variance of X were given by

$$\mu_1(X) = A^2 + 2\psi_0 \quad (3-a)$$

$$\mu_2(X) = 4\psi_0 A^2 + 4\psi_0^2 \quad (3-b)$$

From Equations (3-a) and (3-b) the following moment estimators were found:

$$\hat{A}^2 = +\sqrt{2M'_1(X)^2 - M'_2(X)} \quad (4-a)$$

$$\hat{\psi}_0 = \frac{M'_1(X) - \hat{A}^2}{2} \quad (4-b)$$

where $M'_1(X)$ and $M'_2(X)$ are the sample moments about the origin.

To use any estimator rationally, its properties (e.g., its sampling distribution, expected value, variance) should be known. When the exact sampling distribution is not known but the estimator involves sums of sample values as in this case, standard errors provide a means for determining the large-sample variance of the estimators. The remainder of this report will focus on the properties of the estimators given in Equations (4-a) and (4-b).

III. EXPECTED VALUE AND VARIANCE OF \hat{A}^2 AND $\hat{\psi}_0$

The expected value and the variance of the two estimators can be approximated by expanding \hat{A}^2 and $\hat{\psi}_0$ in a Taylor series about $M'_1(X)$ and $M'_2(X)$ evaluated at $\mu'_1(X)$ and $\mu'_2(X)$ where

$$\mu'_1(X) = E\{M'_1(X)\}$$

$$\mu'_2(X) = E\{M'_2(X)\} \quad .$$

In general, if T is a function of M'_1 and M'_2 then the Taylor series of T expanded around μ'_1 and μ'_2 is

$$\begin{aligned}
T(M'_1, M'_2) &\cong T(\mu'_1, \mu'_2) + \left(\frac{\partial T}{\partial M'_1} \right)_{\substack{M'_1 = \mu'_1 \\ M'_2 = \mu'_2}} (M'_1 - \mu'_1) + \left(\frac{\partial T}{\partial M'_2} \right)_{\substack{M'_1 = \mu'_1 \\ M'_2 = \mu'_2}} (M'_2 - \mu'_2) \\
&+ \frac{1}{2} \left[\left(\frac{\partial^2 T}{\partial M'^2_1} \right)_{\substack{M'_1 = \mu'_1 \\ M'_2 = \mu'_2}} (M'_1 - \mu'_1)^2 + \left(\frac{\partial^2 T}{\partial M'^2_2} \right)_{\substack{M'_1 = \mu'_1 \\ M'_2 = \mu'_2}} (M'_2 - \mu'_2)^2 \right] \\
&+ \left(\frac{\partial^2 T}{\partial M'_1 \partial M'_2} \right)_{\substack{M'_1 = \mu'_1 \\ M'_2 = \mu'_2}} (M'_1 - \mu'_1)(M'_2 - \mu'_2) + \dots \quad (5)
\end{aligned}$$

Taking the expected value of both sides yields

$$\begin{aligned}
E\{T(M'_1, M'_2)\} &\cong E\{T(\mu'_1, \mu'_2)\} + \frac{1}{2} \left[\left(\frac{\partial^2 T}{\partial M'^2_1} \right)_{\substack{M'_1 = \mu'_1 \\ M'_2 = \mu'_2}} \text{var}(M'_1) + \left(\frac{\partial^2 T}{\partial M'^2_2} \right)_{\substack{M'_1 = \mu'_1 \\ M'_2 = \mu'_2}} \text{var}(M'_2) \right] \\
&+ \left(\frac{\partial^2 T}{\partial M'_1 \partial M'_2} \right)_{\substack{M'_1 = \mu'_1 \\ M'_2 = \mu'_2}} \text{cov}(M'_1, M'_2) \quad (6)
\end{aligned}$$

The variances and covariance of M'_1 and M'_2 can be shown to be

$$\text{var}(M'_1) = \frac{1}{\eta} [\mu'_2 - \mu'^2_1] \quad (7-a)$$

$$\text{var}(M'_2) = \frac{1}{\eta} [\mu'_4 - \mu'^2_2] \quad (7-b)$$

$$\text{cov}(M'_1, M'_2) = \frac{1}{\eta} [\mu'_3 - \mu'_1 \mu'_2] \quad (7-c)$$

where η is the sample size.

Therefore, the expected value of T can be written as

$$\begin{aligned}
E\{T(M'_1, M'_2)\} &\cong E\{T(\mu'_1, \mu'_2)\} + \frac{1}{2\eta} \left(\frac{\partial^2 T}{\partial M'^2_1} \right)_{\substack{M'_1 = \mu'_1 \\ M'_2 = \mu'_2}} (\mu'_2 - \mu'^2_1) \\
&+ \left(\frac{\partial^2 T}{\partial M'^2_2} \right)_{\substack{M'_1 = \mu'_1 \\ M'_2 = \mu'_2}} (\mu'_4 - \mu'^2_2) + \frac{1}{\eta} \left[\left(\frac{\partial^2 T}{\partial M'_1 \partial M'_2} \right)_{\substack{M'_1 = \mu'_1 \\ M'_2 = \mu'_2}} (\mu'_3 - \mu'_1 \mu'_2) \right] \quad (8)
\end{aligned}$$

By considering only the first-order terms in Equation (5), the variance of T can also be approximated. Rearranging terms in Equation (5) yields

$$T(M'_1, M'_2) - T(\mu'_1, \mu'_2) \approx \left(\frac{\partial T}{\partial M'_1} \right) (M'_1 - \mu'_1) + \left(\frac{\partial T}{\partial M'_2} \right) (M'_2 - \mu'_2) \quad (9)$$

Squaring both ends and taking the expected value yields

$$\begin{aligned} \text{var} [T(M'_1, M'_2)] &\approx \left(\frac{\partial T}{\partial M'_1} \right)^2 \text{var}(M'_1) + \left(\frac{\partial T}{\partial M'_2} \right)^2 \text{var}(M'_2) \\ &\quad + 2 \left(\frac{\partial T}{\partial M'_1} \right) \left(\frac{\partial T}{\partial M'_2} \right) \text{cov}(M'_1, M'_2) \end{aligned} \quad (10)$$

Therefore, the variance of T can be written as

$$\begin{aligned} \text{var} [T(M'_1, M'_2)] &\approx \frac{1}{n} \left[\left(\frac{\partial T}{\partial M'_1} \right)^2 (\mu'_2 - \mu'_1)^2 + \left(\frac{\partial T}{\partial M'_2} \right)^2 (\mu'_4 - \mu'_2)^2 \right] \\ &\quad + \frac{2}{n} \left[\left(\frac{\partial T}{\partial M'_1} \right) \left(\frac{\partial T}{\partial M'_2} \right) (\mu'_3 - \mu'_1 \mu'_2) \right] \end{aligned} \quad (11)$$

Since the estimators for both \hat{A}^2 and $\hat{\psi}_0$ are functions of M'_1 and M'_2 , using Equations (8) and (11) the expected value and the variance of \hat{A}^2 and $\hat{\psi}_0$ are

$$E\{\hat{A}^2\} \approx A^2 + \frac{1}{2n(2\mu_1^2 - \mu_2^2)^{3/2}} \left[2\mu_1^2 \mu_3^2 - \frac{\mu_4^2}{4} - \frac{7}{4} \mu_2^2 \right] \quad (12-a)$$

$$\text{var}\{\hat{A}^2\} \approx \frac{1}{n(2\mu_1^2 - \mu_2^2)} \left[6\mu_1^2 \mu_2^2 - 4\mu_1^4 + \frac{\mu_4^2}{4} - \frac{\mu_2^2}{4} - 2\mu_1^2 \mu_3^2 \right] \quad (12-b)$$

$$E\{\hat{\psi}_0\} \cong \psi_0 + \frac{1}{2\eta (2\mu_1'^2 - \mu_2')^{3/2}} \left[\frac{7}{8} \mu_2'^2 + \frac{1}{8} \mu_4' - \mu_1' \mu_3' \right] \quad (12-c)$$

$$\begin{aligned} \text{var}\{\hat{\psi}_0\} \cong & \frac{1}{\eta (2\mu_1'^2 - \mu_2')^2} \left\{ \left[\frac{1}{4} (2\mu_1'^2 - \mu_2') - \frac{\mu_1' (2\mu_1'^2 - \mu_2')^{1/2}}{2} \right. \right. \\ & + \mu_1'^2 \left. \right] (\mu_2' - \mu_1')^2 + \frac{(\mu_4' - \mu_2'^2)}{16} + \left[\frac{(2\mu_1'^2 - \mu_2')^{1/2}}{4} \right. \\ & \left. \left. - \frac{\mu_1'}{2} \right] (\mu_3' - \mu_1' \mu_2') \right\} \quad (12-d) \end{aligned}$$

Using Equation (12) and substituting the sample moments for the population moments will yield approximations for any sample taken. Equation (12) can also be expressed in terms of the population parameters A^2 and ψ_0 . That is,

$$E\{\hat{A}^2\} = A^2 + \frac{2\psi_0}{\eta} \left[1 + \frac{\psi_0}{A^2} - \frac{8\psi_0^2}{A^4} - \frac{\psi_0^3}{A^6} \right] \quad (13-a)$$

$$\text{var}\{\hat{A}^2\} = \frac{4\psi_0^2}{\eta} \left[1 + \frac{8\psi_0}{A^2} + \frac{4\psi_0^2}{A^2} \right] \quad (13-b)$$

$$E\{\hat{\psi}_0\} = \psi_0 + \frac{\psi_0}{\eta} \left[\frac{4\psi_0^3}{A^6} + \frac{8\psi_0^2}{A^4} - \frac{\psi_0}{A^2} - 1 \right] \quad (13-c)$$

$$\text{var}\{\hat{\psi}_0\} = \frac{\psi_0^2}{\eta} \left[\frac{2A^2}{\psi_0} + \frac{38\psi_0}{A^2} + \frac{4\psi_0^2}{A^4} + 198 \right] \quad (13-d)$$

As can be seen from Equation (13), \hat{A}^2 and $\hat{\psi}_0$ are consistent estimators in that

- a) The expected value of the estimator approaches the true value as $\eta \rightarrow \infty$.
- b) The variance of the estimator decreases as $\eta \rightarrow \infty$.

The estimators are not unbiased for small sample sizes; however, the bias is negligible for sample sizes greater than 1500 for most parameter values (Tables 1 and 2).

IV. ASYMPTOTIC SAMPLING DISTRIBUTION OF \hat{A}^2 AND $\hat{\psi}_0$

For the normal model, γ_1 (indicator or skewness) is equal to zero and γ_2 (indicator or peakedness) is equal to three; thus, the shape factors of an unknown model can be compared with these to see whether a normal approximation is reasonable. This approach is taken in what follows, to determine the asymptotic properties of the first and second sample moments of Laha's Bessel distribution.

Define

$$X_1 \sim N(a_1, \psi_0)$$

$$X_2 \sim N(a_2, \psi_0)$$

X_1 and X_2 are independent

then

$$R = \sqrt{X_1^2 + X_2^2} \sim \text{Rician distribution}$$

and

$$R^2 = X_1^2 + X_2^2 \sim \text{Laha's Bessel distribution.}$$

R^2 can also be shown to be distributed as a constant times a noncentral chi square distribution

$$X_1^2/\psi_0 \sim \chi_1'^2 \text{ with } v = 1 \text{ and } \lambda_1 = a_1^2/\psi_0$$

$$X_2^2/\psi_0 \sim \chi_2'^2 \text{ with } v = 1 \text{ and } \lambda_2 = a_2^2/\psi_0$$

$$R^2 = \psi_0 \left[\frac{X_1^2}{\psi_0} + \frac{X_2^2}{\psi_0} \right] = \psi_0 \left[\chi_1'^2 + \chi_2'^2 \right]$$

where the term in brackets is a linear combination of noncentral chi squares which is distributed as a noncentral chi square with

$$\begin{aligned}
v &= 2 \\
\lambda &= \lambda_1 + \lambda_2 = \frac{a_1^2 + a_2^2}{2}, \\
\therefore R^2 &= \psi_0 \chi_{v=2}^{\prime 2} \\
&\quad \lambda = a_1^2 + a_2^2 / \psi_0
\end{aligned} \tag{14}$$

The first and second sample moments are denoted by $M_1(R^2)$ and $M_2(R^2)$, respectively. The first and second population moments are denoted by $\mu_1(R^2)$ and $\mu_2(R^2)$, respectively. Now

$$\mu_1(R^2) = \psi_0 \mu_1(\chi_{v,\lambda}^{\prime 2})$$

$$\mu_2(R^2) = \psi_0^2 \mu_2(\chi_{v,\lambda}^{\prime 2})$$

where $\mu_1(\chi_{v,\lambda}^{\prime 2})$ and $\mu_2(\chi_{v,\lambda}^{\prime 2})$ are the first and second population moments of the noncentral chi square.

Substituting sample moments and cumulants the following is obtained:

$$M_1(R^2) = \psi_0 M_1(\chi_{v,\lambda}^{\prime 2}) \tag{15-a}$$

$$M_2(R^2) = \psi_0^2 M_2(\chi_{v,\lambda}^{\prime 2}) \tag{15-b}$$

The objective is to determine the theoretical expression for the first four population moments of the $M_1(R^2)$ and $M_2(R^2)$; that is,

$$\begin{aligned}
&\mu_1[M_1(R^2)] \quad , \quad \mu_2[M_1(R^2)] \quad , \quad \mu_3[M_1(R^2)] \quad , \quad \mu_4[M_1(R^2)] \\
&\mu_1[M_2(R^2)] \quad , \quad \mu_2[M_2(R^2)] \quad , \quad \mu_3[M_2(R^2)] \quad , \quad \mu_4[M_2(R^2)] \quad .
\end{aligned}$$

Population Moments of the First Sample Moment $M_1(R^2)$

$$\mu_1[M_1(R^2)] = \mu_1[\psi_0 M_1(\chi_{v,\lambda}^{\prime 2})] = \psi_0 \mu_1[M_1(\chi_{v,\lambda}^{\prime 2})] \tag{16-a}$$

$$\mu_2[M_1(R^2)] = \psi_0^2 \mu_2[M_1(\chi_{v,\lambda}^{\prime 2})] \tag{16-b}$$

$$\mu_3[M_1(R^2)] = \psi_0^3 \mu_3[M_1(X'_{v,\lambda}^2)] \quad (16-c)$$

$$\mu_4[M_1(R^2)] = \psi_0^4 \mu_4[M_1(X'_{v,\lambda}^2)] \quad (16-d)$$

It can be seen that the problem of determining the population moments of the first sample moment of the distribution of R^2 reduces to finding the population moments of the first sample moment of a noncentral chi square distribution.

The first sample moment of a noncentral chi square distribution is

$$M_1(X'^2) = \frac{1}{\eta} \left\{ \sum_{i=1}^{\eta} X'_{i(v,\lambda)}^2 \right\}$$

and is distributed as a constant $(1/\eta)$ times a noncentral chi square with $v^* = \eta v$ degrees of freedom and noncentrality parameter $\lambda^* = \eta \lambda$; therefore,

$$\mu_1[M_1(R^2)] = \psi_0 \mu_1[M_1(X'^2)] = \frac{\psi_0}{\eta} \mu_1(X'_{v\lambda^*}^2) \quad (17-a)$$

$$\mu_2[M_1(R^2)] = \psi_0^2 \mu_2[M_1(X'^2)] = \frac{\psi_0^2}{\eta} \mu_2(X'_{v\lambda^*}^2) \quad (17-b)$$

$$\mu_3[M_1(R^2)] = \psi_0^3 \mu_3[M_1(X'^2)] = \frac{\psi_0^3}{\eta} \mu_3(X'_{v\lambda^*}^2) \quad (17-c)$$

$$\mu_4[M_1(R^2)] = \psi_0^4 \mu_4[M_1(X'^2)] = \frac{\psi_0^4}{\eta} \mu_4(X'_{v\lambda^*}^2) \quad (17-d)$$

$$\mu_1[M_1(R^2)] = \frac{\psi_0}{\eta} (v^* + \lambda^*)$$

$$\mu_2[M_1(R^2)] = \frac{2\psi_0^2}{\eta} (v^* + 2\lambda^*)$$

$$\mu_3[M_1(R^2)] = \frac{8\psi_0^3}{\eta} (v^* + 3\lambda^*)$$

$$\mu_4[M_1(R^2)] = \frac{12\psi_0^4}{\eta^4} \left\{ 4(v^* + 4\lambda^*) + (v^* + 2\lambda^*)^2 \right\}$$

$$\begin{aligned} \mu_1[M_1(R^2)] &= \frac{\psi_0}{\eta} (\eta v + \eta \lambda) \\ &= \psi_0 (v + \lambda) \\ &= \psi_0 \left(2 + \frac{a_1^2 + a_2^2}{\psi_0} \right) \end{aligned}$$

$$\mu_1[M_1(R^2)] = A^2 + 2\psi_0 \quad (18-a)$$

$$\text{where } A^2 = a_1^2 + a_2^2$$

$$\begin{aligned} \mu_2[M_1(R^2)] &= \frac{2\psi_0^2}{\eta} (\eta v + 2\eta \lambda) \\ &= \frac{2\psi_0^2}{\eta} (v + 2\lambda) \\ &= \frac{2\psi_0^2}{\eta} \left(2 + \frac{2A^2}{\psi_0} \right) \end{aligned}$$

$$\mu_2[M_1(R^2)] = \frac{1}{\eta} (4A^2\psi_0 + 4\psi_0^2) \quad (18-b)$$

$$\begin{aligned} \mu_3[M_1(R^2)] &= \frac{8\psi_0^3}{\eta} (\eta v + 3\eta \lambda) \\ &= \frac{8\psi_0^3}{\eta} (v + 3\lambda) \\ &= \frac{8\psi_0^3}{\eta} \left(2 + \frac{3A^2}{\psi_0} \right) \end{aligned}$$

$$\mu_3[M_1(R^2)] = \frac{1}{\eta^2} \left\{ 24A^2\psi_0^2 + 16\psi_0^3 \right\} \quad (18-c)$$

$$\begin{aligned} \mu_4[M_1(R^2)] &= \frac{12\psi_0^4}{\eta^4} \left\{ 4(\eta v + 4\eta\lambda) + (\eta v + 2\eta\lambda)^2 \right\} \\ &= 12\psi_0^4 \left\{ \frac{4(v + 4\lambda)}{\eta^3} + \frac{(v + 2\lambda)^2}{\eta^2} \right\} \\ &= 12\psi_0^4 \left\{ \frac{4v + 16\lambda}{\eta^3} + \frac{v^2 + 4v\lambda + 4\lambda^2}{\eta^2} \right\} \\ &= 12\psi_0^4 \left\{ \frac{8 + 16A^2/\psi_0}{\eta^3} + \frac{4 + 8A^2/\psi_0 + 4A^4/\psi_0^2}{\eta^2} \right\} \\ &= \frac{48\psi_0^4}{\eta^3} \left\{ 2 + 4A^2/\psi_0 + \eta + 2\eta A^2/\psi_0 + 4\eta A^4/\psi_0^2 \right\} \\ &= \frac{48}{\eta^3} \left\{ 2\psi_0^4 + 4A^2\psi_0^3 + \eta\psi_0^4 + 2\eta A^2\psi_0^3 + \eta A^4\psi_0^2 \right\} \\ \mu_4[M_1(R^2)] &= \frac{48}{\eta^3} \left\{ (\eta + 2)\psi_0^4 + 2(\eta + 2)A^2\psi_0^3 + \eta A^4\psi_0^2 \right\}. \quad (18-d) \end{aligned}$$

The coefficients of skewness and kurtosis can now be determined:

$$\begin{aligned} \gamma_1^2[M_1(R^2)] &= \beta_1[M_1(R^2)] = \frac{\left\{ \mu_3[M_1(R^2)] \right\}^2}{\left\{ \mu_2[M_1(R^2)] \right\}^3} \\ &= \frac{\left\{ \frac{8\psi_0^3}{\eta^2} (v + 3\lambda) \right\}^2}{\left\{ \frac{2\psi_0^2}{\eta} (v + 2\lambda) \right\}^3} \end{aligned}$$

$$\gamma_1^2 [M_1(R^2)] = \frac{8}{\eta} \left\{ \frac{(2 + 3\lambda)^2}{(2 + 2\lambda)^3} \right\} \quad (19-a)$$

$$\begin{aligned} \gamma_2 [M_1(R^2)] &= \beta_2 [M_1(R^2)] = \frac{\mu_4 M_1(R^2)}{\left\{ \mu_2 [M_1(R^2)] \right\}^2} \\ &= \frac{\frac{12\psi_0^4}{\eta^4} \left\{ 4(\eta v + 4\eta\lambda) + (\eta v + 2\eta\lambda)^2 \right\}}{\left\{ \frac{2\psi_0^2}{\eta} (v + 2\lambda) \right\}^2} \\ &= \frac{3}{\eta^2} \left\{ \frac{4\eta v + 16\eta\lambda + \eta^2 v^2 + 4\eta^2 v\lambda + 4\eta^2 \lambda^2}{(v + 2\lambda)^2} \right\} \\ &= \frac{3}{\eta^2} \left\{ \frac{8\eta + 16\eta\lambda + 4\eta^2 + 8\eta^2 \lambda + 4\eta^2 \lambda^2}{(2 + 2\lambda)^2} \right\} \\ &= \frac{3}{\eta^2} \left\{ \frac{2\eta + 4\eta\lambda + \eta^2 + 2\eta^2 \lambda + \eta^2 \lambda^2}{(1 + \lambda)^2} \right\} \\ &= 3 \left\{ \frac{2/\eta + 4\lambda/\eta + 1 + 2\lambda + \lambda^2}{(1 + \lambda)^2} \right\} \end{aligned}$$

$$\gamma_2 [M_1(R^2)] = 3 + \frac{6}{\eta} \left(\frac{1 + 2\lambda}{(1 + \lambda)^2} \right) \quad (19-b)$$

As can be seen from γ_1 and γ_2 , the first sample moment approaches a normal distribution as $\eta \rightarrow \infty$. Table 3 gives some values for β_1 and β_2 for various values of λ .

An alternative approach to determining the population moments of the sample moment $M_1(R^2)$ is by differentiating the characteristic function of the mean of Laha's Bessel variate. The characteristic function is:

TABLE 3. VALUES FOR β_1 AND β_2

$\lambda = 0.25$		0.5		1.0		2.0		8.0	
$\eta =$	β_1	β_2	β_1	β_2	β_1	β_2	β_1	β_2	β_2
10	0.387	3.576	0.363	3.533	0.3125	3.45	0.237	3.333	0.092
20	0.194	3.288	0.181	3.267	0.156	3.225	0.118	3.167	0.046
50	0.077	3.115	0.073	3.107	0.063	3.09	0.047	3.067	0.019
100	0.039	3.058	0.036	3.053	0.031	3.05	0.024	3.033	0.009
200	0.02	3.029	0.018	3.027	0.016	3.02	0.012	3.017	0.005
500	0.008	3.012	0.007	3.011	0.006	3.009	0.005	3.007	0.002
1000	0.004	3.006	0.004	3.005	0.003	3.005	0.002	3.003	0.001
10,000	0.000	3.000	0.000	3.000	0.000	3.000	0.000	3.000	0.000

$$\phi_{\bar{X}}(t) = (1 - it/n\alpha)^{-np} \exp \left\{ \frac{it\beta^2}{4\alpha^2(1 - it/n\alpha)} \right\} [1]$$

where

$$\alpha = \frac{1}{2\psi_0} \quad \beta = \frac{A}{\psi_0} \quad p = 1$$

and n = sample size. The r th moment is given by

$$\mu'_r(\bar{X}) = \frac{d^r \phi_{\bar{X}}(t)/dt^r}{i^r} \Big|_{t=0}$$

The first four moments of $M_1(R^2)$ were found using this method and the results agree with those previously derived.

Population Moments of the Second Sample Moment $M_2(R^2)$

$$\mu_1[M_2(R^2)] = \mu_1[\psi_0^2 M_2(X'_{v,\lambda})] = \psi_0^2 \mu_1[M_2(X'_{v,\lambda})] \quad (20-a)$$

$$\mu_2[M_2(R^2)] = \psi_0^4 \mu_2[M_2(X'_{v,\lambda})] \quad (20-b)$$

$$\mu_3[M_2(R^2)] = \psi_0^6 \mu_3[M_2(X'_{v,\lambda})] \quad (20-c)$$

$$\mu_4[M_2(R^2)] = \psi_0^8 \mu_4[M_2(X'_{v,\lambda})] \quad (20-d)$$

In order to evaluate the preceding expressions, the second sample moment can be expressed in terms of k statistics:

$$M_2(X'_{v,\lambda}) = \frac{\eta - 1}{\eta} k_2(X'_{v,\lambda})$$

where the second sample k statistic is given by

$$k_2(X'_{v,\lambda}) = \frac{1}{\eta} \left\{ \eta \sum_i \left(X'_{\frac{\eta}{6}(v,\lambda)} \right)^2 - \left(\sum_i X'_{i(v,\lambda)} \right)^2 \right\} .$$

Now,

$$\mu_1[M_2(R^2)] = \psi_0^2 \left(\frac{\eta - 1}{\eta} \right) \mu_1[k_2(x'_{v,\lambda})^2] \quad (21-a)$$

$$\mu_2[M_2(R^2)] = \psi_0^4 \left(\frac{\eta - 1}{\eta} \right)^2 \mu_2[k_2(x'_{v,\lambda})^2] \quad (21-b)$$

$$\mu_3[M_2(R^2)] = \psi_0^6 \left(\frac{\eta - 1}{\eta} \right)^3 \mu_3[k_2(x'_{v,\lambda})^2] \quad (21-c)$$

$$\mu_4[M_2(R^2)] = \psi_0^8 \left(\frac{\eta - 1}{\eta} \right)^4 \mu_4[k_2(x'_{v,\lambda})^2] \quad (21-d)$$

The population moments of the k_2 statistics can be expressed in terms of the cumulants of the k_2 statistic.

$$\mu_1[M_2(R^2)] = \psi_0^2 \left(\frac{\eta - 1}{\eta} \right) K_1[k_2(x'_{v,\lambda})^2] \quad (22-a)$$

$$\mu_2[M_2(R^2)] = \psi_0^4 \left(\frac{\eta - 1}{\eta} \right)^2 K_2[k_2(x'_{v,\lambda})^2] \quad (22-b)$$

$$\mu_3[M_2(R^2)] = \psi_0^6 \left(\frac{\eta - 1}{\eta} \right)^3 K_3[k_2(x'_{v,\lambda})^2] \quad (22-c)$$

$$\mu_4[M_2(R^2)] = \psi_0^8 \left(\frac{\eta - 1}{\eta} \right)^4 \left\{ K_4[k_2(x'_{v,\lambda})^2] + 3(K_2[k_2(x'_{v,\lambda})^2])^2 \right\} \quad (22-d)$$

where K is the population cumulant of the k_2 statistic.

Expressions for the first four cumulants for the k_2 statistic can be found in Kendall and Stuart [2]. They are

$$K_1[k_2(x'_{v,\lambda})] = K_2(x'_{v,\lambda}) \quad (23-a)$$

$$K_2[k_2(x'_{v,\lambda})] = \frac{K_4(x'_{v,\lambda})}{\eta} + \frac{2K_2^2(x'_{v,\lambda})}{\eta - 1} \quad (23-b)$$

$$\begin{aligned} \kappa_3[k_2(x'_{v,\lambda})] &= \frac{\kappa_6(x'_{v,\lambda})^2}{\eta^2} + \frac{12\kappa_4(x'_{v,\lambda})^2 \kappa_2(x'_{v,\lambda})}{\eta(\eta-1)} \\ &+ \frac{4(\eta-2) \kappa_3^2(x'_{v,\lambda})}{\eta(\eta-1)^2} + \frac{8\kappa_2^3(x'_{v,\lambda})}{(\eta-1)^2} \end{aligned} \quad (23-c)$$

$$\begin{aligned} \kappa_4[k_2(x'_{v,\lambda})] &= \frac{\kappa_8(x'_{v,\lambda})^2}{\eta^3} + \frac{24\kappa_6(x'_{v,\lambda})^2 \kappa_2(x'_{v,\lambda})}{\eta^2(\eta-1)} \\ &+ \frac{32(\eta-2) \kappa_5(x'_{v,\lambda})^2 \kappa_3(x'_{v,\lambda})}{\eta^2(\eta-1)^2} \\ &+ \frac{8(4\eta^2 - 9\eta + 6) \kappa_4^2(x'_{v,\lambda})}{\eta^2(\eta-1)^3} \\ &+ \frac{144\kappa_4(x'_{v,\lambda})^2 \kappa_2^2(x'_{v,\lambda})}{\eta(\eta-1)^2} \\ &+ \frac{96(\eta-2) \kappa_3^2(x'_{v,\lambda}) \kappa_2(x'_{v,\lambda})}{\eta(\eta-1)^3} + \frac{48\kappa_2^4(x'_{v,\lambda})}{(\eta-1)^3} \end{aligned} \quad (23-d)$$

where

$$\kappa_2(x'_{v,\lambda})^2 = 2(v + 2\lambda) \quad (24-a)$$

$$\kappa_3(x'_{v,\lambda})^2 = 8(v + 3\lambda) \quad (24-b)$$

$$\kappa_4(x'_{v,\lambda})^2 = 48(v + 4\lambda) \quad (24-c)$$

$$\kappa_5(x'_{v,\lambda})^2 = 384(v + 5\lambda) \quad (24-d)$$

$$\kappa_6(x'_{v,\lambda})^2 = 3840(v + 6\lambda) \quad (24-e)$$

$$\kappa_7(x'_{v,\lambda})^2 = 46080(v + 7\lambda) \quad (24-f)$$

$$\kappa_8(x'_{v,\lambda})^2 = 645120(v + 8\lambda) \quad (24-g)$$

$$v = 2 \quad (25-a)$$

$$\lambda = A^2/\psi_0 \quad (25-b)$$

Substituting Equations (11) and (12) into Equations (10) and (9) allows determination of the moments of $M_2(R^2)$ for various values of λ and η . Figures 1 and 2 show the coefficients of skewness and kurtosis for $M_2(R^2)$ for λ and η . As can be seen, the distribution of $M_2(R^2)$ approaches normality as $\eta \rightarrow \infty$ but at a much slower rate than $M_1(R^2)$.

From these results, it is concluded that $M_1(X)$ and $M_2(X)$ have asymptotically normal approximations. But $M_1(X)$ and $M_2(X)$ asymptotically normal implies that any differentiable function of $M_1(X)$ and $M_2(X)$ is also asymptotically normal [3]. Therefore, \hat{A}^2 and $\hat{\psi}_0$, both functions (differentiable) of $M_1(X)$ and $M_2(X)$, are asymptotically normal.

V. COMPARISON OF MOMENT ESTIMATORS TO MAXIMUM LIKELIHOOD ESTIMATORS

Estimation is a common problem in statistics. A meaningful estimator should be chosen from a class of estimators having certain optimal properties (e.g., unbiasedness, minimum variance, consistency, etc.). Some of the better known estimation techniques are as follows:

- a) Maximum likelihood.
- b) BAYES.
- c) MINIMAX.
- d) Method of Moments.
- e) Least squares.

Several of these techniques were considered for determining the estimator of the parameters A^2 and ψ_0 for the Rician distribution [4]. The method of Moments was the only one of the several tried that admitted to explicit form. In previous chapters, the asymptotic properties of these moments estimators were examined. They were shown to be asymptotically normal and consistent.

Because moment estimators are usually relatively inefficient, a study was undertaken to compare them to the maximum likelihood estimators (MLE) for A^2 and ψ_0 . It should be recalled that MLE are said to be

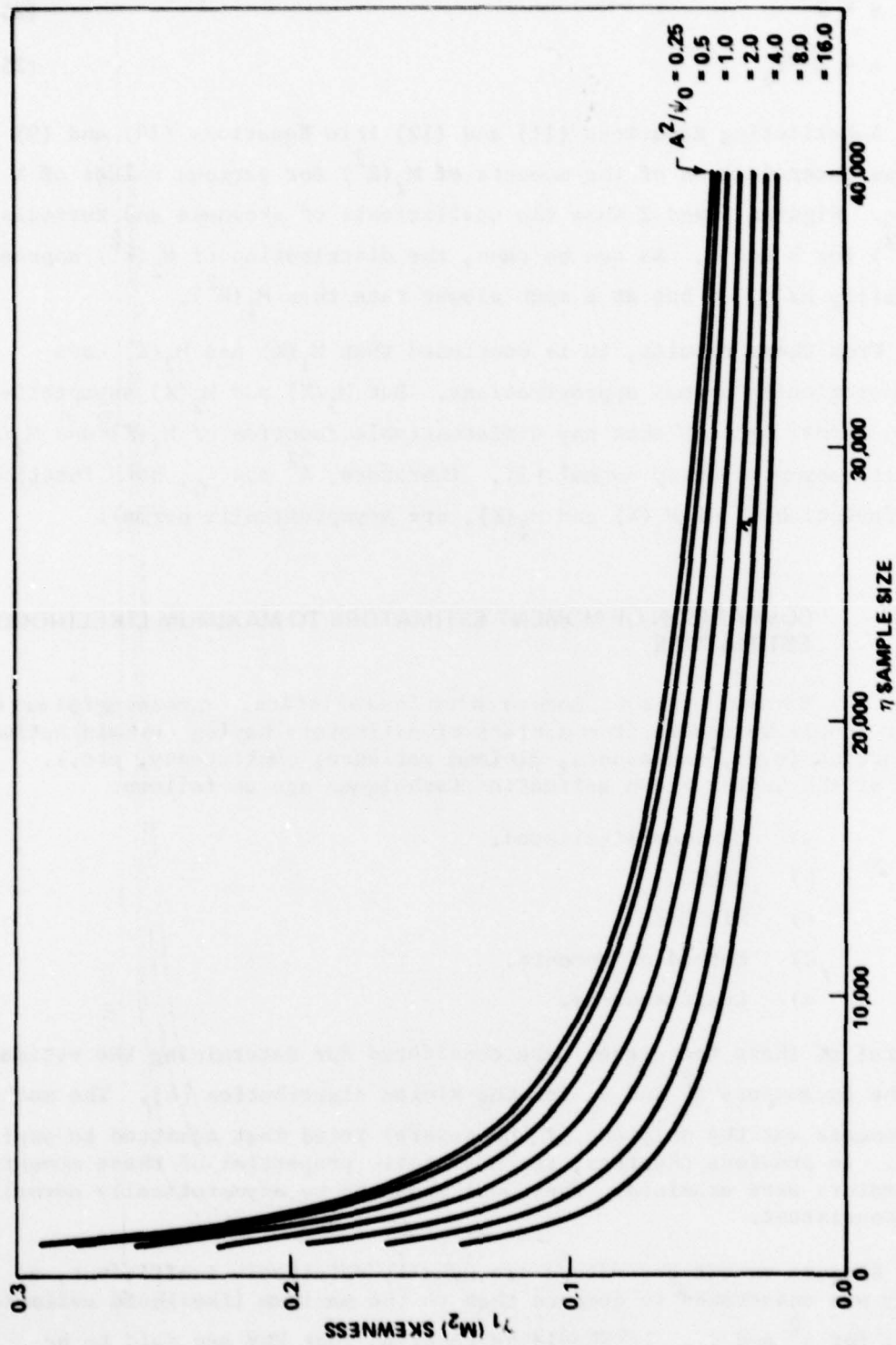


Figure 1. Coefficients of skewness.

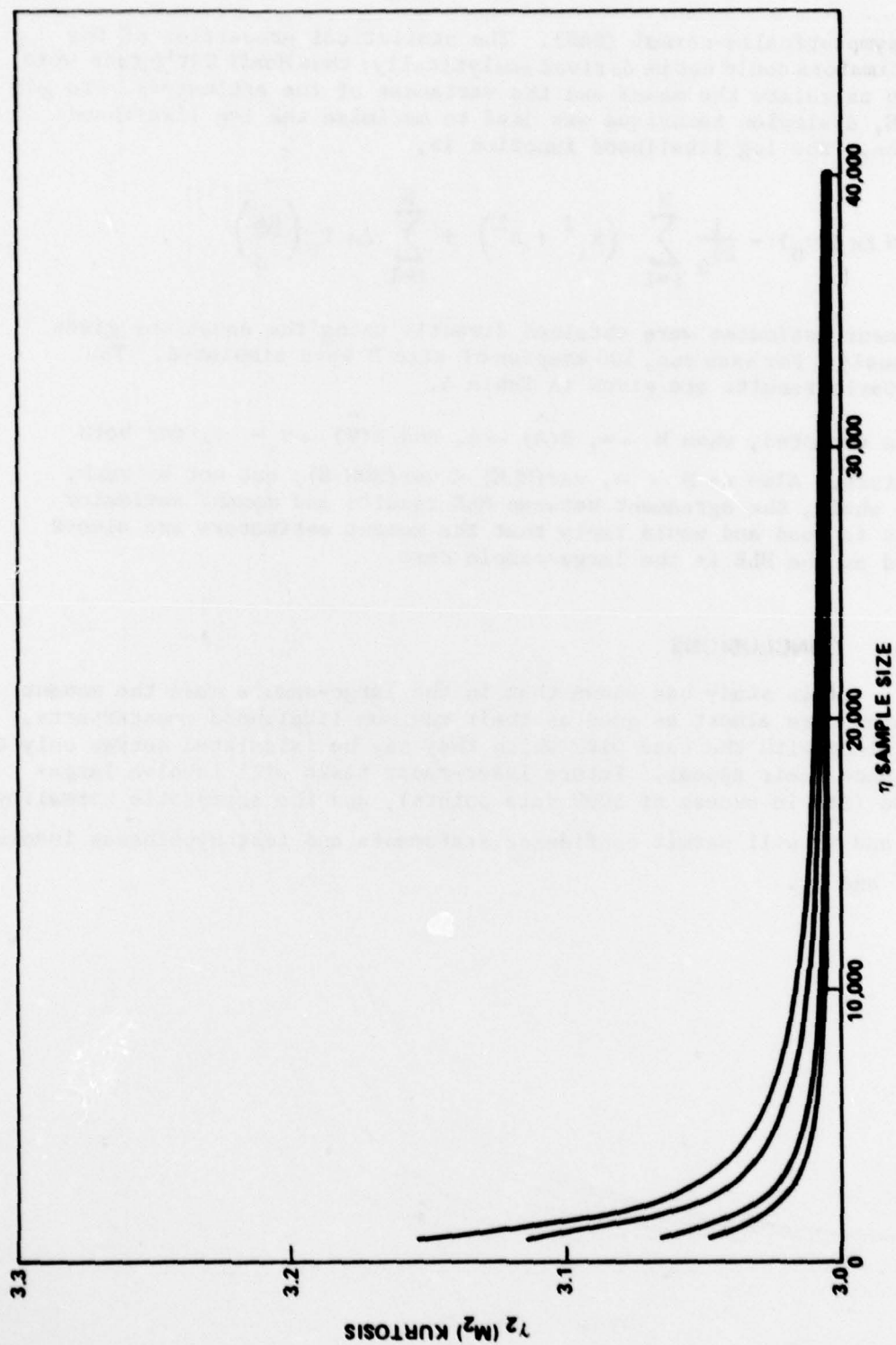


Figure 2. Coefficients of kurtosis.

best asymptotically normal (BAN). The statistical properties of the two estimators could not be derived analytically; thus Monte Carlo runs were made to calculate the means and the variances of the estimators. To get the MLE, a simplex technique was used to maximize the log likelihood function. The log likelihood function is,

$$-N \ell_N(2\psi_0) = -\frac{1}{2\psi_0} \sum_{i=1}^N (R_i^2 + A^2) + \sum_{i=1}^N \ell_N I_0\left(\frac{RA}{\psi_0}\right) .$$

The moment estimates were obtained directly using the equations given previously. For each run, 100 samples of size N were simulated. The Monte Carlo results are given in Table 4.

As expected, when $N \rightarrow \infty$, $E(\hat{A}) \rightarrow A$, and $E(\hat{V}) \rightarrow V = \psi_0$ for both estimators. Also as $N \rightarrow \infty$, $\text{var}(\text{MLE}) < \text{var}(\text{MOM E})$, but not by much. On the whole, the agreement between MLE results and moment estimator results is good and would imply that the moment estimators are almost as good as the MLE in the large-sample case.

VI. CONCLUSIONS

This study has shown that in the large-sample case the moment estimators are almost as good as their maximum likelihood counterparts. This, along with the ease with which they may be calculated serves only to enhance their appeal. Future laser-radar tasks will involve large-samples (far in excess of 5000 data points), and the asymptotic normality of \hat{A}^2 and $\hat{\psi}_0$ will permit confidence statements and test hypotheses involving A^2 and ψ_0 .

TABLE 4. MONTE CARLO RESULTS

	N = 50	N = 100	N = 500
$A = 1.4142$ $\psi_0 = 0.5000$			
AVG (\hat{A}) MLE VAR	1.4379 0.0184	1.4165 0.0107	1.4101 0.0015
AVG (\hat{A}) MOM VAR	1.4237 0.0408	1.4135 0.0147	1.4124 0.0018
AVG (\hat{V}) MLE VAR	0.4870 0.0110	0.4972 0.0067	0.5012 0.0015
AVG (\hat{V}) MOM VAR	0.4945 0.0248	0.5003 0.0109	0.4977 0.0020
$A = 1.4142$ $\psi_0 = 1.0000$			
AVG (\hat{A}) MLE VAR	1.4162 0.1129	1.3973 0.0566	1.4094 0.0079
AVG (\hat{A}) MOM VAR	1.3811 0.1864	1.3649 0.1215	1.4140 0.0085
AVG (\hat{V}) MLE VAR	0.9825 0.1025	1.0059 0.0463	0.9990 0.0111
AVG (\hat{V}) MOM VAR	0.9946 0.1313	1.0167 0.0817	0.9906 0.0129
$A = 1.4142$ $\psi_0 = 2.0000$			
AVG (\hat{A}) MLE VAR	1.1626 0.5271	1.3090 0.2712	1.3699 0.0727
AVG (\hat{A}) MOM VAR	1.3035 0.5973	1.2566 0.4941	1.3711 0.1346
AVG (\hat{V}) MLE VAR	2.1385 0.4153	2.0294 0.2592	2.0238 0.0900
AVG (\hat{V}) MOM VAR	1.9136 0.5623	1.9760 0.3613	1.9860 0.1239

TABLE 4. (Concluded)

	N = 50	N = 100	N = 500
$\Lambda = 2.8284$ $\psi_0 = 0.5000$			
AVG ($\hat{\Lambda}$)	2.8505	2.8348	2.8250
VAR	0.0134	0.0064	0.0010
AVG ($\hat{\Lambda}$)	2.8483	2.8331	2.8254
VAR	0.0137	0.0068	0.0010
AVG (\hat{V})	0.4854	0.4931	0.5011
VAR	0.0079	0.0042	0.0010
AVG (\hat{V})	0.4924	0.4979	0.4999
VAR	0.0103	0.0052	0.0012
$\Lambda = 2.8284$ $\psi_0 = 1.0000$			
AVG ($\hat{\Lambda}$)	2.8621	2.8361	2.8229
VAR	0.0289	0.0147	0.0021
AVG ($\hat{\Lambda}$)	2.8563	2.8337	2.8245
VAR	0.0320	0.0167	0.0025
AVG (\hat{V})	0.9720	0.9916	1.0039
VAR	0.0337	0.0185	0.0044
AVG (\hat{V})	0.9839	0.9963	0.9985
VAR	0.0546	0.0266	0.0059
$\Lambda = 2.8284$ $\psi_0 = 2.0000$			
AVG ($\hat{\Lambda}$) MLE	2.8716	2.8314	2.8192
VAR	0.0721	0.0399	0.0060
AVG ($\hat{\Lambda}$) MOM	2.8475	2.8269	2.8247
VAR	0.1632	0.0586	0.0073
AVG (\hat{V}) MLE	1.9528	2.0133	2.0063
VAR	0.1599	0.0521	0.0231
AVG (\hat{V}) MOM	1.9782	2.0012	1.9910
VAR	0.3972	0.1737	0.0318

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